Acyclic complexes related with noncommutative symmetric functions

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1 Introduction

The algebra of noncommutative symmetric functions **Sym**, introduced in [4], is the free associative algebra (over some field of characteristic zero) generated by an infinite sequence $(S_n)_{n\geq 1}$ of noncommuting indeterminates (corresponding to the complete symmetric functions), endowed with some extra structure imitated from the usual algebra of commutative symmetric functions.

Noncommutative symmetric functions have already been used in different contexts. They provided for instance a simple and unified approach to several topics such as noncommutative continued fractions, Padé approximants and various generalizations of the characteristic polynomial of noncommutative matrices arising in the study of enveloping algebras and their quantum analogues (*cf.* [4] and [11]). Moreover they gave a new point of view towards the classical connections between the free Lie algebra and Solomon's descent algebra (see [3], [6] and [12] for more details). Note also that noncommutative analogues of some aspects of the representation theory of the symmetric group (also related with free Lie algebras) were obtained (*cf.* [8]).

More recently, quantum interpretations of noncommutative symmetric functions and quasi-symmetric functions were even obtained. Indeed it appears that the algebra of noncommutative symmetric functions (resp. of quasi-symmetric functions) is isomorphic to the Grothendieck ring of finitely generated projective (resp. finitely generated) modules over 0-Hecke algebras. Working with the quantum dual point of view, noncommutative ribbon Schur functions and quasi-ribbon functions can be in particular considered as

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cocharacters of irreducible and projective comodules over the crystal limit of Dipper and Donkin's version [1] of the quantum linear group (*cf.* [7] for more details).

In this paper, we present a new aspect of noncommutative symmetric functions. We show how to endow **Sym** with a natural structure of cochain complex which strongly relies on the combinatorics of ribbons. It is interesting to observe that our construction is purely noncommutative: it is not possible to define differentials on ordinary commutative symmetric functions by taking the commutative images of the differentials constructed in this paper. We must however put the stress on the fact that we do not have at this moment any intrinsic interpretation of our complex. According to all the contexts were noncommutative symmetric functions play a role, this complex should certainly have some natural interpretation in the context of Lie algebras or of quantum linear groups. We let as an open problem to the reader to find such an interpretation.

The paper is organized as follows. In Section 2, we briefly present noncommutative symmetric functions (the reader is referred to [4], [6], [2] or [7] for more details on this subject). Section 3 is devoted to the construction of the cochain and chain complexes that are studied in the paper. In Section 4, we give explicit expressions for the images of the corresponding differentials using different classical bases of **Sym**. Finally Section 5 is devoted to the proof of our main result, i.e. the acyclicity of the complexes considered.

2 Noncommutative symmetric functions

The algebra of *noncommutative symmetric functions* is the free associative algebra **Sym** $= \mathbb{Q}\langle S_1, S_2, \ldots \rangle$ generated by an infinite sequence of noncommutative indeterminates S_k, called *complete* functions. For convenience, we set $S_0 = 1$. Let t be another variable commuting with all the S_k. Introducing the generating series

$$\sigma(t) := \sum_{k=0}^{+\infty} S_k\, t^k \ ,$$

one defines other noncommutative symmetric functions by the following relations :

$$\lambda(t) = \sigma(-t)^{-1} \ ,$$

$$\frac{d}{dt}\,\sigma(t) = \sigma(t)\,\psi(t) \ , \quad \sigma(t) = \exp(\phi(t)) \ ,$$

where $\lambda(t)$, $\psi(t)$ and $\phi(t)$ are the generating series given by

$$\lambda(t) := \sum_{k=0}^{+\infty} \Lambda_k\, t^k$$

$$\psi(t) := \sum_{k=1}^{+\infty} \Psi_k\, t^{k-1} \ , \quad \phi(t) := \sum_{k=1}^{+\infty} \frac{\Phi_k}{k}\, t^k \ .$$

The noncommutative symmetric functions Λ_k are called *elementary functions*. On the other hand, Ψ_k and Φ_k are respectively called *power sums* of the *first* and *second kind*.

The algebra **Sym** is graded by the weight function w defined by $w(S_k) = k$. Its homogeneous component of weight n is denoted $\mathbf{Sym}_n$. If (F_n) is a sequence of noncommutative symmetric functions with $F_n \in \mathbf{Sym}_n$ for every $n \geq 1$, then we set

$$F^I = F_{i_1} F_{i_2} \dots F_{i_r}$$

for every composition $I = (i_1, i_2, \dots, i_r)$. The families (S^I), (Λ^I), (Ψ^I) and (Φ^I) all form homogeneous bases of **Sym**.

The set of all compositions of a given integer n is equipped with the *reverse refinement order*, denoted $\prec$. For instance, the compositions J of 4 such that $J \prec (1,2,1)$ are exactly $(1,2,1)$, $(3,1)$, $(1,3)$ and (4). The noncommutative *ribbon Schur functions* (R_I) can then be defined by one of the two equivalent relations

$$S^I = \sum_{J \prec I} R_I \ , \quad R_I = \sum_{J \prec I} (-1)^{\ell(I)-\ell(J)} \, S^J \ ,$$

where $\ell(I)$ denotes the *length* of the composition I, i.e. the number of its parts. One can easily show that the family (R_I) is a homogeneous basis of **Sym**.

The commutative image of a noncommutative symmetric function F is the (commutative) symmetric function f obtained by applying to F the algebra morphism which maps S_n onto h_n, using here the notation of Macdonald [9]. The commutative image of Λ_n is then e_n. On the other hand, Ψ_n and Φ_n are both mapped to p_n. Finally R_I is sent to an ordinary ribbon Schur function, which is usually denoted by r_I.

There is also a strong connection between noncommutative symmetric functions and the descent algebra of the symmetric group. This is the subalgebra of $\mathbb{Q}[\mathfrak{S}_n]$, the group algebra of $\mathfrak{S}_n$, defined as follows. Let us first recall that an integer $i \in [1, n-1]$ is said to be a *descent* of a permutation $\sigma \in \mathfrak{S}_n$ iff $\sigma(i) > \sigma(i+1)$. The *descent set* of a permutation $\sigma \in \mathfrak{S}_n$ is the subset of $[1, n-1]$ that consists of all descents of σ. If $I = (i_1, \dots, i_r)$ is a composition of n, we associate with it the subset $A(I) = \{ d_1, \dots, d_{r-1} \}$ of $[1, n]$ defined by $d_k = i_1 + \dots + i_k$ for every k. We set D_I to be the sum in $\mathbb{Q}[\mathfrak{S}_n]$ of all permutations with descent set $A(I)$. Solomon [13] has shown that the D_I form a linear basis of a subalgebra of $\mathbb{Q}[\mathfrak{S}_n]$ which is called the *descent algebra* of $\mathfrak{S}_n$ and denoted by Σ_n. An isomorphism of graded vector spaces

$$\alpha : \ \Sigma = \bigoplus_{n=0}^{+\infty} \Sigma_n \ \longrightarrow \ \mathbf{Sym} = \bigoplus_{n=0}^{+\infty} \mathbf{Sym}_n$$

is obtained by setting

$$\alpha(D_I) = R_I$$

for every composition I. The direct sum Σ can be endowed with an algebra structure by setting $xy = 0$ for every $x \in \Sigma_p$ and $y \in \Sigma_q$ whenever $p \neq q$. The *internal product*, denoted $*$, on **Sym** is defined by requiring that α be an *anti*-isomorphism, i.e. by setting

$$F * G = \alpha(\alpha^{-1}(G) \, \alpha^{-1}(F)) \ ,$$

for every noncommutative symmetric functions F and G.

The graded dual of **Sym** can also be identified to the algebra **QSym** of *quasi-symmetric functions* introduced by Gessel [5]. Let X be an infinite totally ordered commutative alphabet. Let us then recall that a formal series $f \in \mathbb{Q}[\![X]\!]$ is said to be *quasi-symmetric* iff one has

$$(f|y_1^{i_1}\, y_2^{i_2}\, \dots\, y_k^{i_k}) = (f|z_1^{i_1}\, z_2^{i_2}\, \dots\, z_k^{i_k})$$

for every sequences $y_1 < y_2 < \dots < y_k$ and $z_1 < z_2 < \dots < z_k$ of elements of X and for every exponents $i_1, i_2, \dots, i_k \in \mathbb{N}$. Here, $(f|y)$ stands for the coefficient of y in f. The algebra **QSym** inherits a grading from $\mathbb{Q}[\![X]\!]$. A natural homogeneous basis of **QSym** is then provided by the *quasi-monomial functions* (M_I) defined by

$$M_I = \sum_{y_1 < y_2 < \dots < y_r} y_1^{i_1}\, y_2^{i_2}\, \dots\, y_r^{i_r}$$

for every composition $I = (i_1, i_2, \dots, i_r)$. Another convenient basis is formed by the *quasi-ribbon functions* (F_I) (also called "fundamental" quasi-symmetric functions by Gessel) defined by

$$F_I = \sum_{J \succ I} M_J \ .$$

The pairing $\langle \cdot, \cdot \rangle$ between **QSym** and **Sym** can equivalently be defined by one of the following relations

$$\langle M_I, S^J \rangle = \langle F_I, R_J \rangle = \delta_{I,J}$$

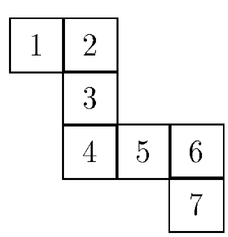
where I, J are arbitrary compositions.

3 Constructions of differentials

3.1 A coboundary operator on Sym

We devote this section to the construction of a coboundary operator on **Sym**. To this purpose, we first need to introduce some notation concerning ribbons and compositions. Let I be a composition of n and let $r(I)$ be the ribbon diagram associated with I. For every integer $i \in [1, n]$, suppressing the i-th box in $r(I)$ (the boxes of $r(I)$ are numbered from left to right and top to bottom by all integers between 1 and n) breaks it into two ribbon diagrams $r(I, i)^-$ and $r(I, i)^+$, in this order. We denote $I^-(i)$ and $I^+(i)$ the compositions of $i-1$ and $n-i$ whose ribbon diagrams are respectively $r(I, i)^-$ and $r(I, i)^+$.

Example 3.1 For the composition $I = (2, 1, 3, 1)$, the corresponding ribbon diagram is (we indicated here the numbering of every box of $r(I)$)



.

Then, e.g., $I^-(3) = (2)$, $I^+(3) = (3, 1)$ and $I^-(4) = (2, 1)$, $I^+(4) = (2, 1)$. □

Let us also recall two natural operations on compositions. If $I = (i_1, \ldots, i_r)$ and $J = (j_1, \ldots, j_s)$ are respectively compositions of n and m, one defines two new compositions $I \cdot J$ and $I \triangleleft J$ by setting

$$I \cdot J = (i_1, \ldots, i_r, j_1, \ldots, j_s) \qquad \text{and} \qquad I \triangleleft J = (i_1, \ldots, i_{r-1}, i_r + j_1, j_2, \ldots, j_s) \ .$$

In other words, $I \cdot J$ and $I \triangleleft J$ are the compositions corresponding to the two different ways of concatenating the ribbon diagram of J at the end of the ribbon diagram of I. The product of two noncommutative ribbon Schur functions can easily be described using these two operations. Indeed, one has

$$R_I \, R_J = R_{I.J} + R_{I \triangleleft J}$$

for any compositions I and J (*cf.* [4]).

We can now define an operator $\delta_n : \mathbf{Sym}_n \longrightarrow \mathbf{Sym}_{n-1}$ by setting

$$\delta_n(R_I) = 2 \, R_{I+(1)} \ + \ \sum_{i=2}^{n-1} \, (-1)^{i-1} \, R_{I-(i)} \, R_{I+(i)} \ + \ (-1)^{n-1} \, 2 \, R_{I-(n)} \ .$$

This leads to a linear operator δ on $\mathbf{Sym}$ defined by

$$\delta = \bigoplus_{n=0}^{+\infty} \ \delta_n \ .$$

The following proposition shows that δ endows $\mathbf{Sym}$ with a structure of cochain complex.

Proposition 3.2 *The family* $\mathbf{R} = (\mathbf{Sym}_n, \delta_n)_{n \geq 0}$ *is a cochain complex.*

Proof — We have to check that $\delta^2 = 0$. To provide a simple proof of this result, we shall now introduce a graphic notation. We use the notation $\boxed{}$ to denote a generic ribbon diagram encoding a noncommutative ribbon Schur function. We also denote by

$$\boxed{} \bullet \boxed{}$$

the sum of the *two* ways of gluing the second ribbon diagram at the end of the first one. Thus this notation encodes the product of two nontrivial ribbon Schur functions. Observe that when one of the ribbon diagrams is empty, both ways of gluing one diagram at the end of the other give the same result. Hence we have

$$\emptyset \bullet \boxed{} = \boxed{} \bullet \emptyset = 2 \ \boxed{} \ .$$

The reader will easily see that the operation $\bullet$ (extended to formal linear combinations of ribbon diagrams) is associative. Before going back to our proof, let us introduce the last piece of notation. If $r(I) = \boxed{}$ is a ribbon diagram, then

$$\boxed{} \overset{i}{\bullet} \boxed{}$$

will stand for $r(I,i)^{-} \bullet r(I,i)^{+}$. We can now rewrite the definition of δ as

$$\delta(\ \boxed{}\ ) = \sum_{i=1}^{n} (-1)^{i-1}\ \boxed{} \overset{i}{\bullet} \boxed{}\ . \tag{1}$$

Therefore

$$\delta(\ \boxed{} \overset{i}{\bullet} \boxed{}\ ) = \sum_{j=1}^{i-1} (-1)^{j-1}\ \boxed{} \overset{j}{\bullet} \boxed{} \overset{i}{\bullet} \boxed{}\ + \sum_{j=i+1}^{n} (-1)^{j}\ \boxed{} \overset{i}{\bullet} \boxed{} \overset{j}{\bullet} \boxed{}$$

for any function of the form $\boxed{} \overset{i}{\bullet} \boxed{}$. Using the two last relations, we obtain

$$\begin{aligned}
&\delta^2(\ \boxed{}\ ) \\
&= \sum_{i=1}^{n}(-1)^{i-1} \Big(\sum_{j=1}^{i-1} (-1)^{j-1}\ \boxed{} \overset{j}{\bullet} \boxed{} \overset{i}{\bullet} \boxed{}\ + \sum_{j=i+1}^{n} (-1)^{j}\ \boxed{} \overset{i}{\bullet} \boxed{} \overset{j}{\bullet} \boxed{}\ \Big) \\
&= \sum_{1 \le j < i \le n} (-1)^{i+j}\ \boxed{} \overset{j}{\bullet} \boxed{} \overset{i}{\bullet} \boxed{}\ - \sum_{1 \le i < j \le n} (-1)^{i+j}\ \boxed{} \overset{i}{\bullet} \boxed{} \overset{j}{\bullet} \boxed{} = 0\ ,
\end{aligned}$$

as desired. □

3.2 First properties

The following proposition gives a compatibility of δ with the multiplication of noncommutative ribbon Schur functions.

Proposition 3.3 *For all compositions I and J, one has*

$$\delta(R_I\, R_J) = \delta(R_I)\, R_J + (-1)^{\ell(I)}\, R_I\, \delta(R_J)\ . \tag{2}$$

Proof — The formula can easily be proved using the graphical formalism introduced in the proof of Proposition 3.2. Using (1), we obtain for every compositions I and J :

$$\begin{aligned}
&\delta(\ \overset{I}{\boxed{}} \bullet \overset{J}{\boxed{}}\ ) \\
&= \sum_{i=1}^{l(I)} (-1)^{i-1}\ \boxed{} \overset{i}{\bullet} \boxed{} \bullet \overset{J}{\boxed{}}\ + \sum_{i=l(I)+1}^{l(I)+l(J)} (-1)^{i-1}\ \overset{I}{\boxed{}} \bullet \boxed{} \overset{i}{\bullet} \boxed{} \\
&= \Big(\sum_{i=1}^{l(I)} (-1)^{i-1}\ \boxed{} \overset{i}{\bullet} \boxed{}\ \Big) \bullet \overset{J}{\boxed{}} + (-1)^{l(I)}\ \overset{I}{\boxed{}} \bullet \Big(\sum_{i=1}^{l(J)} (-1)^{i-1}\ \boxed{} \overset{i}{\bullet} \boxed{}\ \Big) \\
&= \delta(\ \overset{I}{\boxed{}}\ ) \bullet \overset{J}{\boxed{}} + (-1)^{l(I)}\ \overset{I}{\boxed{}} \bullet \delta(\ \overset{I}{\boxed{}}\ )\ ,
\end{aligned}$$

as announced. □

We shall now give some basic symmetry properties of δ. Let us first introduce some notation concerning compositions. If $I = (i_1, \ldots, i_r)$ is a composition, $\overline{I}$ denotes the *mirror image* of I, i.e. the composition $(i_r, \ldots, i_1)$. We also denote by $I^{\sim}$ the composition whose ribbon diagram is the conjugate diagram of the diagram of I. The composition $I = (2, 1, 3, 1)$ of Example 3.1 is for instance self conjugate, i.e. one has

$$I^{\sim} = I = (2, 1, 3, 1) \ ,$$

whereas $I^{\sim} = (2, 1, 2)$ when $I = (1, 3, 1)$. The following result is easy to check.

Proposition 3.4 *For every composition I, one has*

$$\delta(R_{I^{\sim}}) = \delta(R_I)^{\sim} \qquad \text{and} \qquad \delta(R_{\overline{I}}) = (-1)^{l(I)-1}\, \overline{\delta(R_I)} \ . \tag{3}$$

3.3 A decomposition of δ

Let us now introduce two new operators δ_n^+ and δ_n^- from $\mathbf{Sym}_n$ into $\mathbf{Sym}_{n-1}$ by setting

$$\delta_n^+(R_I) = \sum_{i=1}^{n} (-1)^{i-1}\, R_{I^-(i)\cdot I^+(i)} \qquad \text{and} \qquad \delta_n^-(R_I) = \sum_{i=1}^{n} (-1)^{i-1}\, R_{I^-(i) \triangleright I^+(i)} \ ,$$

for every composition I of n. These two operators are associated in an obvious way with the two ways of gluing a ribbon diagram at the end of another. Defining δ^+ and δ^- as the direct sum of the operators δ_n^+ and δ_n^-, we clearly have

$$\delta = \delta^+ + \delta^- \ . \tag{4}$$

Moreover it is easy to see (*cf.* proof of Proposition 3.2) that δ^+ and δ^- are also differentials. We can now summarize our results in the following proposition.

Proposition 3.5 *δ^+ and δ^- are two differentials whose sum is δ.*

This shows that the cochain complex $\mathbf{R}$ can be decomposed into two cochain complexes, denoted $\mathbf{R}^+$ and $\mathbf{R}^-$, naturally associated with the two differentials δ^+ and δ^-.

Note 3.6 Using relation (4) and the fact that δ, δ^+ and δ^- are differentials, one can immediately deduce that $\delta^+ \circ \delta^- + \delta^- \circ \delta^+ = 0$. In fact, a stronger relationship holds between δ^+ and δ^-. The reader can indeed easily check that one has

$$\delta^+ \circ \delta^- = \delta^- \circ \delta^+ = 0 \ .$$

3.4 Dual boundary operators

One can transfer by duality every cochain complex defined on $\mathbf{Sym}$ into a chain complex on $\mathbf{QSym}$. We can thus associate to every operator δ_n a dual operator ∂_n such that the following diagram is commutative (here the $*$ arrows correspond to the natural duality between $\mathbf{Sym}$ and $\mathbf{QSym}$ (*cf.* Section 2)) :

$$\begin{tikzcd} \mathbf{Sym}_{n-1} \arrow[d, "*"'] & \mathbf{Sym}_n \arrow[l, "\delta_n"'] \arrow[d, "*"] \\ \mathbf{QSym}_{n-1} \arrow[r, "\partial_n"'] & \mathbf{QSym}_n \end{tikzcd}$$

This defines a differential ∂ (the direct sum of all operators ∂_n) on the commutative algebra of quasi-symmetric functions. This differential is characterized by

$$\langle \partial(F_I) | R_J \rangle = \langle F_I | \delta(R_J) \rangle \tag{5}$$

for every compositions I and J. We can of course use the same technique to define two others differentials ∂^+ and ∂^- (the dual operators of δ^+ and δ^-) whose sum gives ∂.

4 Differentials in classical bases

In this section we give explicit expressions for the matrices of the differential δ with respect to classical bases of **Sym**. Let us introduce some more notation. If $F = (F_I)$ (resp. $G = (G_I)$) is a basis of $\mathbf{Sym}_n$ (resp. $\mathbf{Sym}_{n-1}$), we denote by $D_n(F,G)$ the $2^{n-1} \times 2^{n-2}$ matrix defined by

$$\delta(F_I) = \sum_{J \vDash n-1} (D_n(F,G))_{IJ}\, G_J \ .$$

We also associate with every $n \times m$ matrix M, the $n \times m$ matrix $\widetilde{M}$ defined by

$$\widetilde{M}_{ij} = M_{n-i+1, m-j+1} \ .$$

It is easy to see that $\widetilde{M}$ is obtained from M by a symmetry with respect to its center. Finally, when M is a $n \times m$ matrix with an even number $n = 2k$ of rows, we denote by $M^{(1)}$ (resp. by $M^{(2)}$) the $k \times m$ submatrix of M formed by the first (resp. last) k rows of M.

4.1 Differentials of ribbon Schur functions

The matrices $D_n(R,R)$ give explicit expressions for the functions $\delta(R_I)$ in the ribbon basis of **Sym**. They can be recursively constructed as follows.

Proposition 4.1 *For every $n \geq 3$, one has*

$$D_n(R,R) = \left(\begin{array}{c|c} A_n & \begin{array}{c} -I_{2^{n-3}} \\ \hline I_{2^{n-3}} \end{array} \\ \hline \begin{array}{c} I_{2^{n-3}} \\ \hline -I_{2^{n-3}} \end{array} & \widetilde{A_n} \end{array} \right)$$

where $I_{2^{n-3}}$ denotes the identity matrix of order 2^{n-3}, and A_n is a $2^{n-2} \times 2^{n-3}$ matrix defined by the following recursive rules

$$A_3 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \; A_4 = \begin{pmatrix} 0 & 1 \\ -1 & 2 \\ 0 & -1 \\ 1 & -2 \end{pmatrix} \quad \textit{and} \quad A_n = \left(\begin{array}{c|c} \begin{array}{c|c} A_{n-2} & \begin{array}{c} -I_{2^{n-5}} \\ \hline I_{2^{n-5}} \end{array} \\ \hline \begin{array}{c} 0_{2^{n-5}} \\ \hline -I_{2^{n-5}} \end{array} & -B_{n-1} \end{array} & \begin{array}{c} I_{2^{n-4}} \\ \hline 2\, I_{2^{n-4}} \end{array} \\ \hline \begin{array}{c} 0_{2^{n-4}} \\ \hline I_{2^{n-4}} \end{array} & B_n \end{array} \right)$$

for every $n \geq 5$, where B_n denotes the $2^{n-3} \times 2^{n-4}$ matrix defined by

$$B_4 = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \quad \textit{and} \quad B_n = \left(\begin{array}{c|c} -\widetilde{B_{n-1}} & \begin{array}{c} -I_{2^{n-5}} \\ \hline 0_{2^{n-5}} \end{array} \\ \hline \begin{array}{c} -I_{2^{n-5}} \\ \hline -I_{2^{n-5}} \end{array} & \begin{array}{c} -B_{n-1}^{(1)} \\ \hline A_{n-2}^{(2)} \end{array} \end{array} \right)$$

for every $n \geq 5$ (the null matrix of order 2^k is always denoted above by 0_{2^k}).

Proof — A thorough analysis of the definition of δ easily gives this result. $\square$

Note 4.2 As an immediate corollary of the last result, the reader may check that all the entries of the matrix $D_n(R, R)$ belong to

- $\{-2, -1, 0, 1, 2\}$ when $n = 2k$ is even;
- $\{-2, -1, 0, 1, 2, 3\}$ when $n = 2k+1$ is odd.

Moreover, the value 3 is only involved in the expansions

$$\delta(R_{2k+1}) = 3\, R_{2k} + \sum_{i=1}^{2k-1} (-1)^i\, R_{i,2k-i-1} \; ,$$

$$\delta(R_{1^{2k+1}}) = 3\, R_{1^{2k}} + \sum_{i=0}^{2k-2} (-1)^{i-1}\, R_{1^i,2,1^{2k-2-i}} \; .$$

Example 4.3 For $n = 2, 3, 4, 5$, the matrices $D_n(R, R)$ are :

$$\begin{array}{cc} & 1 \\ \begin{array}{c} 2 \\ 11 \end{array} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{cc} 2 & 11 \end{array} \\ \begin{array}{c} 3 \\ 21 \\ 12 \\ 111 \end{array} & \begin{pmatrix} 3 & -1 \\ 1 & 1 \\ 1 & 1 \\ -1 & 3 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{cccc} 3 & 21 & 12 & 111 \end{array} \\ \begin{array}{c} 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{array} & \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \end{array},$$

$$
\begin{array}{c|c}
 & \begin{array}{cccccccc} 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \end{array} \\
\begin{array}{c} 5 \\ 41 \\ 32 \\ 311 \\ 23 \\ 221 \\ 212 \\ 2111 \\ 14 \\ 131 \\ 122 \\ 1211 \\ 113 \\ 1121 \\ 1112 \\ 11111 \end{array} &
\begin{pmatrix}
3 & -1 & 1 & 0 & -1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 0 & -1 & 0 \\
-1 & 2 & 0 & 2 & 0 & 0 & 0 & -1 \\
0 & 0 & 2 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & -1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 2 & 0 & 0 \\
-1 & 0 & 0 & 0 & 2 & 0 & 2 & -1 \\
0 & -1 & 0 & 0 & 0 & 2 & 1 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & -1 & 0 & 1 & -1 & 3
\end{pmatrix}
\end{array} .
$$

□

Let us now denote by $D_n^+(R,R)$ (resp. $D_n^-(R,R)$) the matrix of the differential δ^+ (resp. δ^-), restricted to $\mathbf{Sym}_n$, in the ribbon basis. Observe first that

$$D_n(R,R) = D_n^+(R,R) + D_n^-(R,R)$$

for $n \geq 1$. The matrices $D_n^-(R,R)$ and $D_n^+(R,R)$ can be computed from $D_n(R,R)$ using the following result.

Proposition 4.4 *For every $n \geq 1$, one has*

$$D_n^+(R,R) = D_n^-\widetilde{(R,R)} \ .$$

Proof — A simple analysis of the nature of the central symmetry involved in our formula shows that our proposition follows from the fact that $(I \cdot J)^\sim = J^\sim \triangleleft I^\sim$ for every compositions I and J. □

4.2 Differentials of complete functions

The δ-images of complete functions are given by the matrices $D_n(S,S)$. These can be recursively constructed using the following proposition in conjunction with the first several matrices $D_n(S,S)$ given in Example 4.7.

Proposition 4.5 *For every $n \geq 5$, the matrix $D_n(S,S)$ is equal to*

$$\left(\begin{array}{c|c}
\begin{array}{c|c}
\begin{array}{c} D_{n-2}(S,S)^{(1)} \\ \hline \begin{array}{c|c} 0_{2^{n-4}\times 2^{n-5}} & -D_{n-3}(S,S) \end{array} \end{array} & \begin{array}{c} I_{2^{n-4}} \\ \hline 4\,I_{2^{n-4}} \end{array} \\ \hline
0_{2^{n-3}\times 2^{n-4}} & D_{n-2}(S,S)
\end{array} & \begin{array}{c} -\,I_{2^{n-3}} \\ \hline 0_{2^{n-3}} \end{array} \\ \hline
\begin{array}{c} 4\,I_{2^{n-3}} \\ \hline 0_{2^{n-3}} \end{array} & \begin{array}{c} -D_{n-1}(S,S)^{(1)} \\ \hline \begin{array}{c|c} 0_{2^{n-3}\times 2^{n-4}} & D_{n-2}(S,S) \end{array} \end{array}
\end{array}\right) .$$

Proof — Using Proposition 4.1, one can easily show that

$$\delta(S_{2n}) = \sum_{i=1}^{2n-2} (-1)^{i+1}\, S^{2n-1-i,i} \;,$$

$$\delta(S_{2n+1}) = 4\,S_{2n} + \sum_{i=1}^{2n-1} (-1)^{i}\, S^{2n-i,i} \;,$$

for every n. An adapted version of Proposition 3.3, together with a thorough analysis, gives the required result. $\square$

Corollary 4.6 *Every entry of $D_n(S,S)$ belongs to $\{-4,-1,0,1,4\}$.*

Example 4.7 For $n = 2,3,4,5$, the matrices $D_n(S,S)$ are :

$$\begin{array}{c} \\ 2 \\ 11 \end{array}\!\!\begin{array}{c} 1 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{array} , \qquad
\begin{array}{c} \\ 3 \\ 21 \\ 12 \\ 111 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 2 & 11 \end{array} \\ \begin{pmatrix} 4 & -1 \\ 4 & 0 \\ 4 & 0 \\ 0 & 4 \end{pmatrix} \end{array} , \qquad
\begin{array}{c} \\ 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 3 & 21 & 12 & 111 \end{array} \\ \begin{pmatrix} 0 & 1 & -1 & 0 \\ -4 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & -4 & 1 \\ 0 & 4 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} ,$$

$$
\begin{array}{c}
 \\ 5 \\ 41 \\ 32 \\ 311 \\ 23 \\ 221 \\ 212 \\ 2111 \\ 14 \\ 131 \\ 122 \\ 1211 \\ 113 \\ 1121 \\ 1112 \\ 11111
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \end{array} \\
\begin{pmatrix}
4 & -1 & 1 & 0 & -1 & 0 & 0 & 0 \\
4 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & -1 \\
0 & 0 & 4 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 4 & 0 & 0 & 4 & -4 & 0 & 1 \\
0 & 0 & 4 & 0 & 0 & 0 & 4 & -1 \\
0 & 0 & 0 & 4 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4
\end{pmatrix}
\end{array} .
$$

□

5 Cocycles and coboundaries

This section is devoted to the proof of the acyclicity of the different complexes constructed above. We also give explicit bases for the cochain (or coboundary) modules of these complexes. For every $n \geq 1$, let us define recursively the subset $I(n)$ of the set of all compositions of n as follows

$$
I(n) = \begin{cases} \{ (1) \} & \text{if } n = 1, \\ \emptyset & \text{if } n = 2, \\ \{ (i_1, \ldots, i_r) \,|\, (i_1 = 2 \text{ and } (i_2, \ldots, i_r) \in I(n-2)) \text{ or } (i_1 \geq 3) \} & \text{if } n \geq 3 \ . \end{cases}
$$

It can be easily checked that $I(n)$ consists of all compositions of n that are greater than or equal to (in the sense of the lexicographic order) $2^k\,1$ when $n = 2k+1$ and to $2^{(k-2)}\,3\,1$ when $n = 2k$. Let us also denote by $i(n)$ the number of compositions in $I(n)$. By construction, $i(n)$ satisfies to the recurrence relations

$$
i(1) = 1, \ i(2) = 0 \ \text{ and } \ i(n) = 2^{n-3} + i(n-2) \ \text{ for } \ n \geq 3 \ . \tag{6}
$$

It follows that

$$
i(n) = \frac{2^{n-1} + 2\,(-1)^{n-1}}{3} \tag{7}
$$

for every $n \geq 1$. One can also give the generating series of the sequence $i(n)$:

$$
\sum_{n=1}^{\infty} i(n)\, t^n = \frac{t\,(1-t)}{(1+t)\,(1-2t)} \ . \tag{8}
$$

We are now in a position to prove our main result.

Proposition 5.1 *The cochain complex* $\mathbf{R} = (\mathbf{Sym}_n, \delta_n)_{n\geq 0}$ *is acyclic.*

Proof — We first prove by induction on n that the family $(\delta(R_I))_{I\in I(n)}$ is linearly independent. This is clear for $n = 1$ and $n = 2$. Let us suppose now that $n \geq 3$. Observe that one has

$$R_2\, R_J = R_{2,J} + R_{2+j_1,J'}$$

for every composition $J = (j_1)\cdot J'$ of $n-2$. Since $2+j_1 \geq 3$, it follows from this identity and from the definition of $I(n)$ that the independence of the family $(\delta(R_I))_{I\in I(n)}$ is equivalent to the independence of the family consisting of all $\delta(R_J)$, with $J = (j_1,\ldots,j_r)$ and $j_1 \geq 3$, and all $\delta(R_2\, R_J)$, with $J \in I(n-2)$. Proposition 4.1 implies that

$$\delta(R_J) = -R_{1,j_1-2,j_2,\ldots,j_r} + \sum_{H\prec(1,j_1-2,j_2,\ldots,j_r)} c_H\, R_H$$

for every composition $J = (j_1,\ldots,j_r)$ with $j_1 \geq 3$. On the other hand, using Propositions 3.3 and 4.1, one can write

$$\delta(R_2\, R_J) = R_2\, \delta(R_J) = \pm\, R_{2,H} + \sum_{L\prec(2,H)} c_L\, R_L\ ,$$

for every composition $J \in I(n-2)$. It now suffices to use the independence of the family $(\delta(R_J))_{J\in I(n-2)}$ and a simple triangularity argument to deduce the independence of the family $(\delta(R_I))_{I\in I(n)}$ from the two last above relations. It follows that

$$\dim\, \mathrm{Im}\, \delta_n \geq |I(n)| = i(n)\ ,$$

for every $n \geq 0$. On the other hand,

$$\dim\, \mathrm{Ker}\, \delta_n = \dim\, \mathbf{Sym}_n - \dim\, \mathrm{Im}\, \delta_n \leq 2^{n-1} - i(n) = i(n+1)\ ,$$

for every $n \geq 1$. But, since $\delta^2 = 0$, one always has $\mathrm{Im}\, \delta_n \subset \mathrm{Ker}\, \delta_{n-1}$. Hence

$$\dim\, \mathrm{Im}\, \delta_n \leq \dim\, \mathrm{Ker}\, \delta_{n-1} \leq i(n)\ .$$

Thus $\dim\, \mathrm{Im}\, \delta_n = i(n)$, and

$$\dim\, \mathrm{Ker}\, \delta_{n-1} = \dim\, \mathbf{Sym}_{n-1} - \dim\, \mathrm{Im}\, \delta_{n-1} = 2^{n-2} - i(n-1) = i(n)\ .$$

This shows that $\mathrm{Ker}\, \delta_{n-1}$ and $\mathrm{Im}\, \delta_n$ have the same dimensions, implying

$$\mathrm{Im}\, \delta_n = \mathrm{Ker}\, \delta_{n-1}\ ,$$

for every $n \geq 1$ as desired. □

The results obtained in the proof of Proposition 5.1 have the following immediate consequence.

Corollary 5.2 *For every* $n \geq 1$*, the family* $(\delta(R_I))_{I\in I(n)}$ *is a basis of the cochain (or coboundary) module* $\mathrm{Im}\, \delta_n = \mathrm{Ker}\, \delta_{n-1}$*, which has dimension* $i(n)$*.*

Corollary 5.3 *For every $n \geq 1$, the family $(\delta(S^I))_{I \in I(n)}$ is a basis of the cochain (or coboundary) module* $\mathrm{Im}\,\delta_n = \mathrm{Ker}\,\delta_{n-1}$.

Proof — According to Corollary 5.2, it is sufficient to check linear independence of the family $(\delta(S^I))_{I \in I(n)}$. This follows from Proposition 4.5. □

Corollary 5.4 *The dual chain complex* $\mathbf{R}^* = (\mathbf{QSym}_n, \partial_n)_{n \geq 0}$ *is acyclic.*

Note 5.5 One can also dualize Corollaries 5.2 and 5.3 to obtain explicit bases for the chain (or boundary) modules associated with $\mathbf{R}^*$.

Using techniques similar to those of the proof of Proposition 5.1 together with Proposition 4.4, it is possible to obtain the following result.

Proposition 5.6 *The cochain complexes* $\mathbf{R}^+ = (\mathbf{Sym}_n, \delta_n^+)_{n \geq 0}$ *and* $\mathbf{R}^- = (\mathbf{Sym}_n, \delta_n^-)_{n \geq 0}$ *are acyclic.*

Corresponding versions of Corollaries 5.2 and 5.3 are also valid for the complexes $\mathbf{R}^+$ and $\mathbf{R}^-$.

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